# Triangles on Cones: Solution

The relationship between the sum of the vertex angles $A$, $B$, and $C$ of a triangle with a cone point in its interior and the cone point angle $X$ is simply that $A+B+C+X = 540°$ (or $3\pi$ radians). This can be discovered experimentally, by making cones of various cone point angles, drawing triangles on those cones around the cone points, measuring the vertex angles, and summing (and recognizing slight measurement errors). The proof, however, may involve a different approach. Here is one such.

First suppose we are on the surface of a (flat) Euclidean plane, a.k.a. a 360° cone. Let us take the perspective of a small car making a single circuit of our triangle (it doesn't matter whether the circuit is clockwise or counterclockwise, as long as we keep track of changes to orientation). In making the three turns necessary at each vertex to stay on the "road" (the triangle), the driver eventually rotates the car a full 360° cone by the time the car returns to its original starting point. See Figure 1 for a sketch. If the three vertex angles are labeled as $A$, $B$, and $C$, then the angles by which the car changes its orientation at each turn are $180° - A$, $180° - B$, and $180° - C$. Therefore we can write the equation

$$(180° - A) + (180° - B) + (180° - C) = 360°,$$

which we can simplify algebraically to

$$A + B + C = 180°,$$

thus proving the well-known formula for the usual sum of the vertex angles of a triangle.

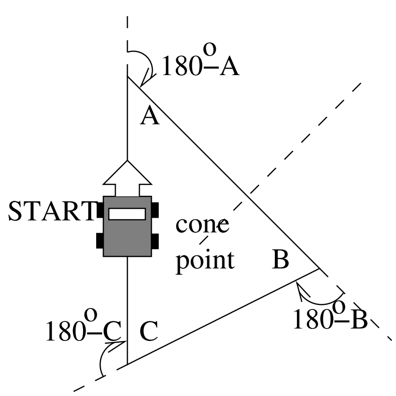

FIGURE 1. On a plane.

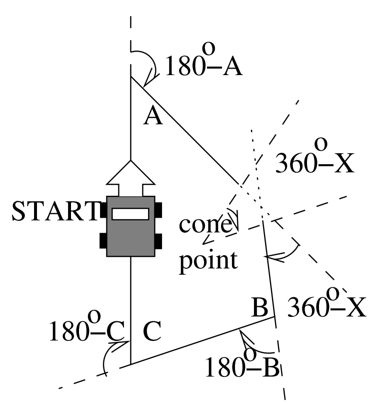

FIGURE 2. On a cone.

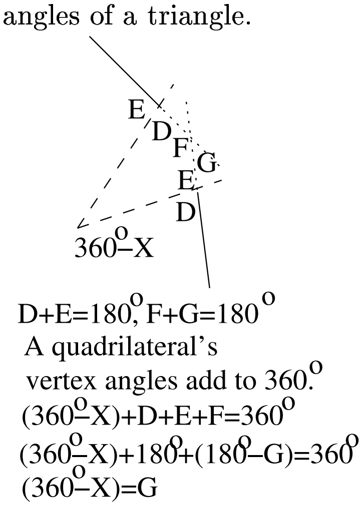

Figure 3. Expanded view.

If, instead, we are on the surface of a cone whose cone point is other than 360° (in Figure 2 it is depicted as being less), then in making a complete circuit of the triangle, the car actually goes through four orientation changes if we cut open the cone along a "seam" (as depicted): the three turns made at the triangle's vertices, plus another one made upon crossing the seam. If the cone point angle is $X$, then the "defect" (shown in Figure 2 as an empty section of space) has an angle of $360° - X$, and by identifying supplementary angles (see Figure 3) we can see that the change in the car's orientation in crossing the seam is also $360° - X$. Thus we have four orientation changes (instead of three) which result in one full rotation of the car's orientation. We write

$$(180° - A) + (180° - B) + (180° - C) + (360° - X) = 360°,$$

which we can simplify algebraically to

$$540° - A - B - C + (360° - X) = 360°,$$

or, finally,

$$(A + B + C) + X = 540°. \quad \square$$