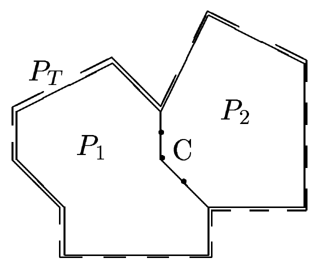
Proof of Pick's Theorem

This proof has three parts. First: we show that it is *additive*: that is, if Pick's Theorem applies to two polygons P_1 and P_2 which share one or more edges, then it also applies to the larger polygon P_T which is made by putting P_1 and P_2 together (and removing the common edges). Second, we show that any polygon on a geoboard can be constructed by adding in this way (and possibly subtracting!) a finite set of right triangles with their legs parallel to the geoboard axes. Third and finally, we show that Pick's Theorem holds for such triangles.

Additivity

Suppose that we have two polygons P_1 and P_2 which share one or more edges. Let P_T be the polygon formed by joining P_1 and P_2 and removing their common edges. Let C be the number of geoboard points that fall on those common edges, excluding the two endpoints (which are geoboard points by assumption, since Pick's Theorem applies to geoboard polygons only).



Now, if we use subscripts 1, 2, and T to distinguish the area A, number of interior geoboard points I, and number of boundary geoboard points B for each polygon, then we can calculate

$$\begin{aligned} A_T &= A_1 + A_2 \\ I_T &= I_1 + I_2 + C \\ B_T &= B_1 + B_2 - 2C - 2 \end{aligned}$$

since area is additive, the points interior to P_T are either interior to P_1 or P_2 or else on their shared boundary C, and the boundary points of P_T are the boundary points of P_1 without C together with the boundary points of P_2 without C (the -2 is to avoid double-counting the two endpoints of the boundary common to P_1 and P_2 but not P_T). Now, if Pick's Theorem holds for P_1 and P_2, then

$$A_1 = I_1 + \frac{B_1}{2} - 1 \text{ and } A_2 = I_2 + \frac{B_2}{2} - 1,$$

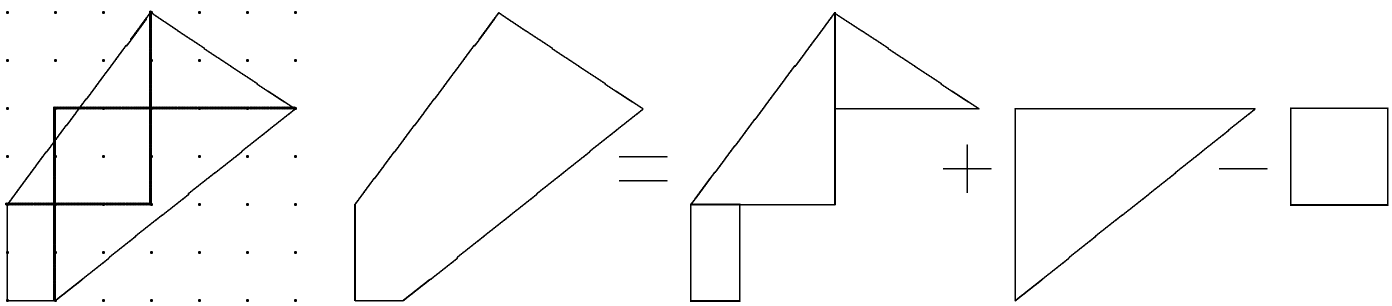
and

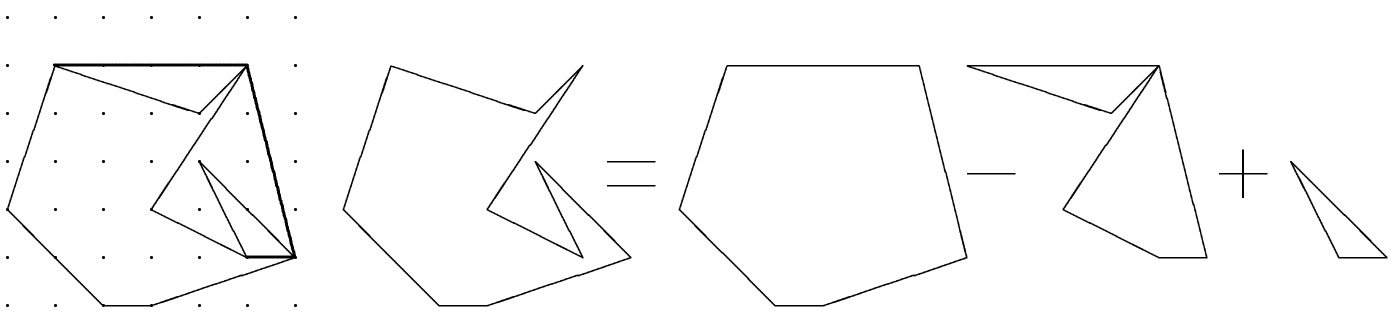
$$\begin{aligned} A_T &= A_1 + A_2 \\ &= \left(I_1 + \frac{B_1}{2} - 1\right) + \left(I_2 + \frac{B_2}{2} - 1\right) \\ &= (I_1 + I_2 + C) + \left(\frac{B_1}{2} + \frac{B_2}{2} - C - 1\right) - 1 \\ &= I_T + \frac{B_T}{2} - 1, \end{aligned}$$

so Pick's Theorem holds for P_T as well. Thus the formula is indeed additive in this sense.

Geoboard polygons from right triangles

Since the vertices of geoboard polygons are geoboard points, every edge of a geoboard polygon is the hypotenuse of some right triangle whose other (shorter) edges are parallel to the geoboard axes. We can then divide up any convex polygon into a combination (including "negatives" where there is overlap) of such right triangles (and possibly rectangles, which can be considered pairs of such right triangles, matched along the diagonal), and any concave polygon can be considered a convex polygon "minus" another polygon which fills in the gap. Both of these notions are illustrated below:





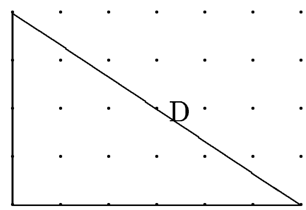
Pick's Theorem for rectangles and right triangles

Consider a geoboard rectangle with its sides parallel to the geoboard axes. If it has length L and width W, then it encompasses a grid of points with dimension $L+1$ by $W+1$. (For example, a 1-by-2 rectangle is drawn along a 2-by-3 grid of points.) For the rectangle, we then compute

$$\begin{aligned} A &= L \times W \\ I &= (L-1) \times (W-1) \\ B &= 2L + 2W \\ \text{so } I + \frac{B}{2} - 1 &= (L-1)(W-1) + \frac{2L+2W}{2} - 1 = LW = A \end{aligned}$$

which proves Pick's Theorem for such rectangles.

Now let us consider right triangles formed by cutting such rectangles in half along a diagonal. In general, the hypotenuse (the diagonal of the rectangle) may pass through D geoboard points in addition to the two that form its endpoints. (In the diagram below $D = 1$.)



We calculate

$$
\begin{aligned}
A =& \frac{1}{2} L \times W \\
I =& \frac{1}{2}[(L-1) \times (W-1) - D] \\
B =& L + W + 1 + D
\end{aligned}
$$

since the area of the triangle is half that of the rectangle, the number of points interior to the triangle is half that of the points interior to the rectangle and not on the hypotenuse, and the points on the boundary of the triangle are the $L + W + 1$ along the two short sides of the triangle plus the D on the hypotenuse (besides the endpoints). Finally, we calculate

$$I + \frac{B}{2} - 1 = \frac{1}{2}[(L-1) \times (W-1) - D] + \frac{L + W + 1 + D}{2} - 1 = \frac{1}{2} L \times W = A,$$

so Pick's Theorem holds for these right triangles as well.

For any geoboard polygon, we now write it as a composition of right triangles like this and apply additivity to see that Pick's Theorem holds for it as well. Done!